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RESEARCH REPORT

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Note on Stochastic Calculus for Stable Processes

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Note on Stochastic Calculus for Stable Processes I

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Contents

Introduction

If only I had the theorems! Then I should find the proofs easily enough.
B.Riemman

The following text provides an overview on important definitions and theorems from stochastic calculus with special focus on the class of $\alpha-$stable processes on one-dimensional real line.

In the first part of this text is introduced the class of infinitely divisible distributions and stated Lèvy-Khinchine formula. Stable distributions as an representant of this class are defined. Further, there is described algorithm for generating random variables from stable distributions.

The second part provides a quick excursion into definition of Lèvy process and stable process. The link to self-simmilar processes is described.

The third section is focused on the understanding of path properties and mainly jump structure of Lèvy processes. Firstly, there is introduced Lèvy measure as the intensity of the Poisson process, given definition of Poisson random measure and described integration of measurable function with respect to Poisson random measure. Finally there is formulated and explained Lèvy-Itô decomposition.

Fourth section uses the results of Lèvy-Itô decomposition and provides an insight into quadratic variation for Lèvy processes and mainly pure jump processes. Then there is formulated change of variable formula and provided example of its application to stochastic exponential driven by Lèvy processes. The special example for stochastic exponential driven by $\alpha-$stable Lèvy motion is formulated and described simulation technique of such a particular example.

The fifth and last section is devoted to problematics of change of measure and building Lèvy process via this technique. The section contains two theorems which holds for Lèvy processes in general. These are further applied on the subclass of $\alpha-$stable Lèvy process and the result of that investigation is formulated into Theorem 5.3.

1 Stable Distributions

1.1 Infinitely Divisible Distributions

Let us consider a probability measure μ on $\mathbb{R}$ and its characteristic function given by

$$\hat{\mu}(k) = \int_{\mathbb{R}} e^{ikx} \mu(\mathrm{d}x), \text{ where } k \in \mathbb{R}$$

Further we denote by μ^n the n-fold convolution probability measure μ with itself, i.e.

$$\mu^n = \underbrace{\mu * \ldots * \mu}_{\text{n}}$$

Definition 1.1 *A probability measure μ on $\mathbb{R}$ is* infinitely divisible *if, for any $n \in \mathbb{N}$, there exists a probability measure μ_n on $\mathbb{R}$ such that $\mu = \mu_n^n$.*

The convolution of measures is equivalent to the product of their characteristic funtions. This gives us idea about how to verify whether the probability distribution is infinitely divisible. Having the distribution μ, we find the nth root of its characteristic function $\hat{\mu}(k)$ and check if it can be choosen as the characteristic function of some probability measure.

For $n = 0$, the '0th' root of μ is δ_0, a Dirac measure with mass at 0.

The simplest examples of infinitely divisible distributions are Poisson and Gamma distributions, Dirac point masses, Gaussian and stable distributions. On the other hand, uniformly distributed random variables or binomial random variables are not infinitely divisible.

The set of infinitely divisible distributions form an Abelian group with respest to convolution.

Characteristic functions of infinitely divisible distributions are well described by Lèvy-Khinchine formula. This formula provides a representation of the characteristic exponent. This beautifull result is a cornerstone of the whole theory.

Theorem 1.1 *(Lèvy-Khintchine formula)*
Let μ is an infinitely divisible distribution on $\mathbb{R}$ with characteristic exponent $\psi(k)$, i.e.

$$\hat{\mu}(k) = \int_{\mathbb{R}} e^{ikx} \mu(\mathrm{d}x) = e^{-\psi(k)},\ k \in \mathbb{R}$$

then

1. $$\psi(k) = -i\gamma k + \frac{\sigma^2 k^2}{2} - \int_{\{|x| \geq 1\}} (e^{ikx} - 1)\nu(\mathrm{d}x) - \int_{\{|x|<1\}} (e^{ikx} - 1 - ikx)\nu(\mathrm{d}x) \tag{1}$$

   where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure on $\mathbb{R}$ satisfying

   $$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}} \min(|x|^2, 1)\nu(\mathrm{d}x) < \infty \tag{2}$$

2. *The representation of $\hat{\mu}(x)$ in (1) by σ, ν and γ is unique.*

3. *Conversely, if $\sigma \geq 0$, ν is measure satisfying conditions in (2), and $\gamma \in \mathbb{R}$, then there exists an infinitely divisible distribution μ whose characteristic exponent $\psi(k)$ is given by (1).*

Definition 1.2 *We call (σ, ν, γ) from Theorem 1.1 the* generating triplet *of infinitely divisible distribution μ. ν is called* Lèvy measure *of μ and σ Gaussian component of distribution μ.*

To condsider simple examples, Gaussian distribution with mean γ and variance σ^2 has generating triplet $(\sigma, 0, \gamma)$, Poisson distribution with parameter λ has generating triplet $(0, 0, \lambda\delta_1)$. To Dirac mass at point z corresponds triplet $(0, 0, z)$.

The integral

$$\int_{\mathbb{R}} (e^{ikx} - 1 - ikx)\nu(\mathrm{d}x)$$

is integrable, because it is bounded outside neighbourhood of 0 and for fixed k

$$e^{ikx} - 1 - ikx1_{\{0<|x|<1\}})\nu(\mathrm{d}x) \text{ as } |x| \to 0$$

There other ways for getting integrability by choosing correctly the *centering function*.

Definition 1.3 *Let $c : \mathbb{R} \to \mathbb{R}$ is bounded and measurable function, satisfying*

$$\int_{\mathbb{R}} (e^{ikx} - 1 - ikc(x))\nu(\mathrm{d}x) < \infty \text{ for any fixed } k$$

We call $c(x)$ the centering function.

Then we can reformulate Lèvy-Khinchine formula as

$$\psi(k) = -i\gamma_c k + \frac{\sigma^2 k^2}{2} - \int_{\mathbb{R}} (e^{ikx} - 1 - ikxc(x))\nu(\mathrm{d}x)$$

with

$$\gamma_c = \gamma + \int_{\mathbb{R}} x(c(x) - 1_{\{0<|x|<1\}})\nu(\mathrm{d}x)$$

and obtain characteristic triplet (σ, ν, γ_c) for infinititely divisible measure μ. which corresponds to different parametrization.

Obviously sufficient requirement on $c(x)$ is

$$c(x) = 1 + o(|x|) \text{ as } |x| \to 0$$
$$c(x) = O\Big(\frac{1}{|x|}\Big) \text{ as } |x| \to \infty$$

The following choices of $c(x)$ are sometimes used

$$c(x) = 1_{\{0<|x|\leq\varepsilon\}}(x) \text{ with } \varepsilon > 0$$
$$c(x) = \frac{1}{1+|x|^2}$$
$$c(x) = 1_{\{0<|x|\leq 1\}} + \frac{\mathrm{sgn}(x)}{x} 1_{\{1<|x|\}}$$
$$c(x) = \frac{\sin(x)}{x}$$

The centering function obviously affects the representation of the γ term in the formula. For this reason, one has to be carefull with choosing different parametrizations and the choice should depend on the form of Lèvy measure.

1.2 Stable Distributions

Stable distributions belongs to the class of infinitly divisible distributions. Their characteristic exponent can be represented by Lèvy-Khinchine formula. There are other possible ways of representations, see e.g. Zolotarev [6]. In the following, we firstly give definition of the the stable measure and then state theorem which gives most commonly used represantation of characteristic exponent of stable random variable.

Definition 1.4 *Let μ is infinitely divisible probability measure on $\mathbb{R}$. It is called* stable *if, for any $a > 0$, there exist $b > 0$ and $c \in \mathbb{R}$ s.t.*

$$\hat{\mu}(k)^a = \hat{\mu}(bk)e^{ick}$$

It is called strictly stable *if, for any $a > 0$, there is $b > 0$ s.t.*

$$\hat{\mu}(k)^a = \hat{\mu}(bk)$$

We define the stable ditribution by its characteristic function which uniquely determines the form of it.

The Lèvy measure for a real valued stable variable is expressed as

$$\nu(\mathrm{d}x) = \begin{cases} \frac{c_1}{x^{\alpha+1}} 1_{\{x>0\}} + \frac{c_2}{|x|^{\alpha+1}} 1_{\{x<0\}} & \text{for } 0 < \alpha < 2 \text{ and } c_1, c_2 \geq 0, c_1 + c_2 > 0 \\ 0 & \text{for } \alpha = 2 \end{cases}$$

We see that for $\alpha = 2$ we have Gaussian distribution. For $0 < \alpha < 2$ the generating triplet is $(0, \nu, \gamma)$ where Lèvy measure is written above. According the Lèvy-Khinchine formula the characteristic exponent of the stable distribution can be rewriten as

$$\psi(k) = i\gamma k + c_1 \int_0^\infty (e^{ikx} - 1 - \frac{ikx}{1+x^2}) \frac{1}{x^{1+\alpha}} (\mathrm{d}x) + \tag{3}$$

$$+ c_2 \int_{-\infty}^0 (e^{ikx} - 1 - \frac{ikx}{1+x^2}) \frac{1}{|x|^{1+\alpha}} (\mathrm{d}x) \tag{4}$$

Note, that we here we used centering function $c(x) = \frac{1}{1+|x|^2}$ The representation in the following theorem is often used as the definition of $\alpha-$stable law.

Theorem 1.2 *Let $0 < \alpha < 2$ and μ is non-trivial $\alpha-$stable measure, then*

$$\psi(k) = \begin{cases} \sigma|k|^\alpha (1 - i\beta \operatorname{sgn}(k) \tan(\frac{\pi\alpha}{2})) - i\gamma k & \text{for } \alpha \neq 1 \\ \sigma|k|(1 + i\beta(\frac{2}{\pi} \operatorname{sgn}(k) \log|k|) - i\gamma k & \text{for } \alpha = 1 \end{cases} \tag{5}$$

with $\sigma > 0, \beta \in [-1,1]$ and $\gamma \in \mathbb{R}$. Here σ, β and γ are uniquely determined by μ. Conversely, for every $\sigma > 0, \beta \in [-1,1]$ and $\gamma \in \mathbb{R}$, there is non-trivial $\alpha-$stable ditribution μ satisfying (5). A necessary and sufficient condition for a non-trivial $\alpha-$stable distribution to be strictly $\alpha-$stable is that $\gamma = 0$ or that $\beta = 0$, according as $\alpha \neq 1$ or $\alpha = 1$.

The parameters from the previous theorem has the following meaning: $\alpha \in (0,2]$ is called *stability* parameter, $\beta \in [-1,1]$ is *skewness* parameter, $\sigma > 0$ *scale* parameter and $\gamma \in \mathbb{R}$ corresponds to *shift* parameter.

The parameter β represents non-symmetry of the Lèvy measure. ν is symmetric only if $\beta = 0$. For $\beta = 1$ the support of the measure is the positive part of real line, for $\beta = -1$ is the Lèvy measure concentrated only on negative part of real line. Other useful link between Lèvy measure ν and skewness parameter β is that $\beta = \frac{c_1 - c_2}{c_1 + c_2}$ where $c_1, c_2 \geq 0, c_1 + c_2 > 0$ are constants from the Lèvy measure.

If random variable Z has $\alpha-$stable distribution with parameters σ, β and γ we use notation $Z \sim S_\alpha(\sigma, \beta, \gamma)$. For symetric stable random variable $Z \sim S_\alpha(1,0,0)$ we use shorter notation $Z \sim S_\alpha S$

1.3 Simulating from Stable Distribution

The densities for $\alpha-$stable processes are in general not known in closed form. The only known densities are for Gaussian, Cauchy and Lèvy distribution. The following theorem provides us with an algoritm for simulating from stable distributions.

Theorem 1.3 *Let V is uniformly distributed random variable on the interval* $(-\frac{\pi}{2},\frac{\pi}{2})$*, W is exponentialy distributed random variable with mean 1, V and W are independent random variables and* $\alpha \in (0,2]$.

For any $\beta \in [1,1]$ *and* $\alpha \neq 1$*, define* $\theta_0 = \frac{\arctan(\beta\tan(\pi\alpha/2))}{\alpha}$.

Then random variable Z defined by

$$Z = \begin{cases} \frac{\sin\alpha(\theta_0+V)}{(\cos\alpha\theta_0\cos V)^{1/\alpha}}\left[\frac{\cos(\alpha\theta_0+(\alpha-1)V)}{W}\right]^{\frac{1-\alpha}{\alpha}} & \alpha \neq 1 \\ \frac{2}{\pi}\left[(\frac{\pi}{2}+\beta V)\tan V - \beta\log\left(\frac{\frac{\pi}{2}W\cos V}{\frac{\pi}{2}+\beta V}\right)\right] & \alpha = 1 \end{cases}$$

has $S_\alpha(1,\beta,0)$ *distribution.*

Remark 1.1 *In case of symmetric stable distribution, i.e. for* $\beta = 0$ *the formula from previous theorem can be considerably simplified. Following the notation from Theorem 1.3, the random variable*

$$Z = \begin{cases} \frac{\sin(\alpha V)}{(\cos V)^{1/\alpha}}\left[\frac{\cos((\alpha-1)V)}{W}\right]^{\frac{1-\alpha}{\alpha}} & \alpha \neq 1 \\ \tan V & \alpha = 1 \end{cases}$$

has $S_\alpha S$ *distribution.*

Let us take a closer look on the number θ_0 from Theorem 1.3. Consider random random variable Z with $S_\alpha(\sigma,\beta,0)$ distribution. Define $\rho = P(Z \geq 0)$. In Zolotarev [6], section 2.6 is shown that the probability of $\alpha-$stable random variable having non-negative value depends only on the stability parameter α and the skewness parameter β and can be computed for $\alpha \neq 1,2$ as

$$\rho = \frac{1}{2} + \frac{\arctan(\beta\tan(\pi\alpha/2))}{\pi\alpha}.$$

ρ is called positivity parameter. Observe, that ρ does not depend on the scaling parameter σ. For $0<\alpha<1$, ρ ranges over interval $[0,1]$, whereas for $1<\alpha<2$, ρ ranges over $[1-\frac{1}{\alpha},\frac{1}{\alpha}]$. The boundary points $\rho = 0$ for $\alpha \in (0,1)$ and $\rho = 1-\frac{1}{\alpha}$ for $\alpha \in (1,2)$, respectively, corresponds to the situation that random variable Z takes negative values only. Analogically, the boundary points $\rho = 1$ for $\alpha \in (0,1)$ and $\rho = \frac{1}{\alpha}$ for $\alpha \in (1,2)$ corresponds to the situation that Z has non-negative values only. For symmetric stable distributions, i.e. for $\beta = 0$, the positivity parameter ρ has value $\frac{1}{2}$. We see that $\rho = \frac{1}{2}+\frac{\theta_0}{\pi}$ where θ_0 contains the relation between skewness and stability parameter.

2 Stable processes

2.1 Lèvy processes

Stable processes form a subclass of more general class of Lèvy processes. Assume we are on filtered probability space $(\Omega,\mathcal{F},\mathbb{F},P)$. Let us recall definition of the Lèvy process .

Definition 2.1 *An adapted process* $X = \{X(t), t\geq 0\}$ *is a* Lèvy process *if*

1. *X has increments independent of the past; i.e.*
   $X(t)-X(s)$ *is independent of the* $\mathcal{F}_s$, $0 \leq s < t < \infty$

2. *X has stationary increments, i.e.*
   $X(t) - X(s)$ *has same distribution as* $X(t-s)$ *for* $0 \le s < t < \infty$

3. *X(t) is continuous in probability, i.e.*

$$\lim_{t \to s} P(\omega \in \Omega : |X(\omega,t) - X(\omega,s)| > \varepsilon) = 0$$

For each $t > 0$ the distribution of Lèvy process $X(t)$ is infinitely divisible. It can be shown also that for each infinitely divisible probability measure μ there exists Lèvy process X s.t. μ is distribution of $X(1)$. The following theorem gives us idea about important path property of Lèvy processes.

Theorem 2.1 *Let X be a Lèvy process. There exists a unique modification Y of X which is righ contiuous and has limits from left (càdlàg) and which is also a Lèvy process .*

2.2 $\alpha-$stable Lèvy Motion

Consider filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and on this space consider Lèvy process $L_{\alpha,\beta} = \{L_{\alpha,\beta}(t), t \ge 0\}$. In the following we will call the stochastic process $L_{\alpha,\beta} = \{L_{\alpha,\beta}(t), t \ge 0\}$ $\alpha-$*stable Lèvy motion* if

1. $L_{\alpha,\beta}(0) = 0 \; P-a.s.$

2. $L_{\alpha,\beta}(t)$ has independent increments

3. $L_{\alpha,\beta}(t) - L_{\alpha,\beta}(s) \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$ for any $0 \le s < t < \infty$

Simply said, $L_{\alpha,\beta} = \{L_{\alpha,\beta}(t), t \ge 0\}$ is a Lèvy process if $L_{\alpha,\beta}(t)$ has α-stable distribution. Only the scale parameter changes during time, the stability and scale parameteres does not depend on time and remains constant for all $t > 0$.

Definition 2.2 $X = \{X(t), t \ge 0\}$ *is a Lèvy process on* $\mathbb{R}$. *It is called* stable *or* strictly stable *if the distribution of* $X(1)$ *is, respectively, stable or strictly stable.*

Definition 2.3 $X = \{X(t), t \ge 0\}$ *is a stochastic process on* $\mathbb{R}$. *It is called* selfsimilar *if, for any* $a > 0$, *there exists* $b > 0$ *s.t.*

$$\{X(at), t \ge 0\} \stackrel{d}{=} \{bX(t), t \ge 0\}$$

It is called broad-sense selfsimilar *if, for any* $a > 0$, *there exists* $b > 0$ *and maping* $c : \mathbb{R}^+ \to \mathbb{R}$ *s.t.*

$$\{X(at), t \ge 0\} \stackrel{d}{=} \{bX(t) + c(t), t \ge 0\}$$

The property of selfsimilarity means that when performing scaling changes in the time domain of the process one has to count with a scaling effect in a spatial domain of the process. The broad-sense selfsimilarity is slightly generous concept. Here, when we scale time of the process, the resulting change for the value of the process corresponds to composite mapping where the value of the stochastic process without time change is linearly transformed to capture the time scaling.

The corespondence between stable and selfsimilar Lèvy processes is very straightforward. Having Lèvy process $X = \{X(t), t \ge 0\}$ on $\mathbb{R}$, it is selfsimilar if and only if it is a strictly stable process. Analogically, for broad-sense selfsimilar Lèvy process and stable process. From the Definition 2.3 is obvious, that selfsimilar or broad-sense selfsimilar process does not have to be Lèvy process . If, however, it is a Lèvy process , then it can be only strictly stable or stable process. The class of stable processes is thus intersection of Lèvy processes and broad-sense selfsimilar processes.

3 Poisson Random Measures

Let us start with introducing the concept of Poisson random measures which is crucial for good understanding of behaving of the jump structure of Lèvy process .

Consider Lèvy process $X = \{X(t), t \geq 0\}$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Denote by $X(t-) = \lim_{s \to t^-} X(s)$ and by $\Delta X(t) := X(t) - X(t-)$ the jump of the process X at time t; theorem 2.1 guarrantes that the trajectory of the Lèvy process is càdlag. Consider borel subset $U \in \mathcal{B}(\mathbb{R} \setminus 0)$, s.t. $0 \notin \bar{U}$ where $\bar{U}$ is the closure of set U, i.e. we consider set U to be bounded away from 0. For $t > 0$ define:

$$N(t,U) := \sum_{0 \leq s \leq t} 1_U(\Delta X(s)),$$

that is the number of jumps of size $\Delta X(s) \in U, s \in [0,t]$ which occurs up to time t. As U is bounded away from 0, $N(\cdot)$ really counts jumps of the process. Let us further define random variables

$$T_1^U = \inf\{t > 0 : \Delta X(t) \in U\}$$
$$\vdots$$
$$T_{n+1}^U = \inf\{t > T_n^U : \Delta X(t) \in U\}$$

The sequence $\{T_i^U\}, i = 1, 2, \ldots$ is the sequence of random times in which the jumps are of size U at maximum. The sequence of times of repeated entrances into U is obviously a stopping time as $\{t \leq T_n^U\} \in \mathcal{F}_{t+} = \mathcal{F}_t$ (X has càdlag trajectories and filtration is complete and right-continuous).

We can further rewrite the variable $N(t,U)$ in terms of number of times of jump events which occur up till time t:

$$N(t,U) := \sum_{0 \leq s \leq t} 1_U(\Delta X(s)) = \sum_{n=1}^{\infty} 1_{\{T_n^U \leq t\}} \tag{6}$$

It can be argued that $N(t,U)$ is a counting process, without explosion, with stationary and independent increments which directly implies that $N(t,U)$ is a Poisson process; see Protter [4], p.26.

Let us define Lèvy measure $\nu(U)$ of process X in terms of expected number of jumps of size in U over time unit:

$$\nu(U) := \mathbb{E}[N(1,U)],\ U \in \mathcal{B}(\mathbb{R} \setminus 0).$$

Lèvy measure $\nu(U)$ is the intensity of the Poisson process $N(t,U)$ and is finite because Poisson process has bounded jumps and every Lèvy process with bounded jumps has finite moments of all orders; see Protter [4], Theorem 34, p.25.

Mapping $U \to N(t,U)$ defines a $\sigma-$finite meausure on U. Let us summerize the above into the definition of Poisson random measure.

Definition 3.1 *Consider probability space* $(\Omega, \mathcal{F}, P)$ *and* $\sigma-$*finite measurable space* $(E, \mathcal{E}, \mu)$. *A family of non-negative integer valued random variables* $\{N(U), U \in \mathcal{E}\}$ *is called a* Poisson random measure *on E with intensity* μ, *if the following hold:*

1. *for every* U, $N(U)$ *has Poisson distribution with mean* $\mu(U)$.
2. *if* $U_1, \ldots, U_n$ *are disjoint, then* $N(U_1), \ldots, N(U_n)$ *are independent*
3. *for every* $\omega \in \Omega$, $N(\cdot, \omega)$ *is a measure on* $(E, \mathcal{E})$

Remark 3.1 *1. $N(t,U)$ is a Poisson random measure with intensity measure $\nu(U)$; see Sato [5], Theorem 19.2 (i), p.120.*

2. $\tilde{N}(\cdot,\cdot)$ denotes compensated jump measure defined by $\tilde{N}(t,U) := N(t,U) - t\nu(U)$. It is easy to check that $\mathbb{E}\tilde{N}(t,U) = 0$.

3.1 Integrals with respect to Poisson Random Measures

Consider borel measurable function f which is finite on the set $U \in \mathcal{B}(\mathbb{R}\setminus 0)$. Then its very natural to the sum of jumps of size in U mapped by f, up till time t.

$$I(t,U) = \int_U f(x)N(t,\cdot,dx) = \sum_{0<s\leq t} f(\Delta X(s))1_U(\Delta X(s))$$

Process $I(U) = \{I(t,U), t \geq 0\}$ is again a Lèvy process. For specific choice of $f(x) = x$ we obtain

$$J(t,U) = \int_U xN(t,\cdot,dx) = \sum_{0<s\leq t} \Delta X(s)1_U(\Delta X(s))$$

Process $J(t,U)$ is called *associated jump process* and it is sum of jumps in U up till time t. The process $Y = \{Y(t) = X(t) - J(t,U), t \geq 0\}$ will remain also Lèvy process. See Protter [4], Theorem 37, p.27.

Choose $U = \mathbb{R}\setminus(0,1)$, then process Y is the Lèvy process without big jumps (jumps bigger then 1),

$$Y(1,t) = X(t) - J(t,\mathbb{R}\setminus(0,1)) = X(t) - \int_{|x|\geq 1} xN(t,\cdot,dx)$$

Let us state theorem from Protter [4], Theorem 38, p.28 which describes behaving of the Lèvy measure in terms of expactation of the integral with respect to Poisson random measure.

Theorem 3.1 *Let U be a Borel set with $0 \notin \bar{U}$. Let ν be the Lèvy measure of X, and let $f1_U \in \mathcal{L}(\mathrm{d}\nu)$. Then*

$$\mathbb{E}\Big(\int_U f(x)N(t,\cdot,\mathrm{d}x)\Big) = t\int_U f(x)\nu(\mathrm{d}x)$$

If further $f1_U \in \mathcal{L}^2(\mathrm{d}\nu)$ then

$$\mathbb{E}\Big(\Big[\int_U f(x)N(t,\cdot,\mathrm{d}x) - t\int_U f(x)\nu(\mathrm{d}x)\Big]^2\Big) = t\int_U f(x)^2\nu(\mathrm{d}x)$$

Another important property of associated jump process is its behaving on the disjoint sets. Consider two disjoint Borel sets U_1, U_2 bounded away from 0. Consider processes

$$J(t,U_1) = \sum_{0<s\leq t} \Delta X(s)1_{U_1}(\Delta X(s))$$

and

$$J(t,U_2) = \sum_{0<s\leq t} \Delta X(s)1_{U_2}(\Delta X(s))$$

These will be independent Lèvy processes.

The following two theorems provide view on the Lèvy processes as semimartingales, see Protter [4] Theorem 40, 41, p.30. The first theorem tells us that we can decompose the Lèvy process into martingale and finite variation process. The second theorem gives us idea about construction of the martingale for Lèvy process with bounded jumps. For the Lèvy process with bounded jumps, we can construct the martingale by compensating the original Lèvy process with its expected value and this martingale can be decomposed into the continuous part and jump part. The continuous part of the martingale is a Brownian motion, the jump part of the martingale is infinite sum of compensated Poisson processes. These two new processes are again Lèvy processes and are independent.

Theorem 3.2 *Let $X = \{X(t), t \geq 0\}$ is a Lèvy process . Then $X(t) = V(t) + M(t)$, where V, M are Lèvy processes, V has paths of finite variation and M is a martingale with bounded jumps, i.e. M is process with finite moments of all orders.*

Theorem 3.3 *Let $X = \{X(t), t \geq 0\}$ be a Lèvy process with bounded jumps by a, i.e $\sup_{0<s\leq t} |X(s)| \leq a$ a.s. Let $M(t) = X(t) - \mathbb{E}X(t)$. Then M is a martingale and $M(t) = M(t)^c + M(t)^d$ where $M(t)^c$ is a martingale with continuous trajectories and M^d is a martingale*

$$M^d(t) = \int_{\{|x|\leq a\}} x(N(t,\cdot,\mathrm{d}x) - t\nu(\mathrm{d}x)) = \int_{\{|x|\leq a\}} x(\tilde{N}(t,\cdot,\mathrm{d}x)$$

M^c and M^d are independent Lèvy processes.

3.2 Lèvy-Itô Decomposition

Using the results from previous subsection, we state a very beautifull result which provides us with clear insight into the properties of trajectories of Lèvy process .

Theorem 3.4 *Let $X = \{X(t), t \geq 0\}$ is Lèvy process on $\mathbb{R}$ with generating triplet (σ, ν, γ) and jump measure $N(t,\cdot,\mathrm{d}x)$ of process X, i.e. $N(t,\cdot,\mathrm{d}x)$ is a Poisson random measure. Then the process X can be decomposed into three mutually independent Lèvy processes, for all $t \geq 0$, $P-$a.s.:*

$$X(t) = X^1(t) + X^2(t) + X^3(t)$$

such that

$$X^1(t) = \gamma t + \sigma^2 W(t) \quad \text{where } W(t) \text{ denotes Wiener process} \tag{7}$$

$$X^2(t) = \lim_{\varepsilon\to 0^+} \int_0^t \int_{\{\varepsilon<|x|<1\}} x(N(\mathrm{d}s,\cdot,\mathrm{d}x) - \nu(\mathrm{d}x)\,\mathrm{d}s) \tag{8}$$

$$X^3(t) = \int_0^t \int_{\{1\leq|x|<\infty\}} xN(\mathrm{d}s,\cdot,\mathrm{d}x) \tag{9}$$

The convergence in $X^2(t)$ is uniform in t on any bounded interval.

The Lèvy-Itô decomposition allows us to decompose any Lèvy process into three independent processes, the drifted Brownian motion, process of big jumps and a martingale process of small jumps. In the following we describe the conection with Lèvy-Khinchine formula. Recall first the form of characteristic exponent of distribution of increments of Lèvy process:

$$\psi(k) = -i\gamma k + \frac{\sigma^2}{2} - \int\limits_{\{|x|\geq 1\}} (e^{ikx} - 1)\nu(\mathrm{d}x) - \int\limits_{\{|x|<1\}} (e^{ikx} - 1 - ikx)\nu(\mathrm{d}x) \tag{10}$$

The first part of the formula $\psi_1(k) = -i\gamma k + \frac{\sigma^2}{2}$ clearly corresponds to characteristic function of the distribution of linearly drifted Brownian motion. That is exactly process $X^1(t)$ in the Lèvy-Itô decomposition.

The second part of the formula $\psi_2(k) = \int_{\{|x|\geq 1\}}(e^{ikx}-1)\nu(\mathrm{d}x)$ is the characteristic function of the compound Poisson process with intensity $\nu(\mathbb{R}\setminus(-1,1))$ and size of jumps distributed according law $\frac{\nu(dx)}{\nu(\mathbb{R}\setminus(-1,1))}$ where Lèvy measure is supported on $\mathbb{R}\setminus(-1,1)$. It corresponds to process $X^3(t)$, i.e. the part of big jumps is driven by compound Poisson process with intesity characterized by the size of the real line without unit ball centered in origin measured by Lèvy measure and distribution of the jumps corresponds to Lèvy measure standartized by the intesity of driving Poisson process.

The last part of the formula $\psi_3(k) = \int\limits_{\{|x|<1\}}(e^{ikx}-1-ikx)\nu(\mathrm{d}x)$ is an infinite sum of drifted compound Poisson processes with different intensities and distributions. To see it, denote by $\lambda_n = \nu(\{x; 2^{-(n+1)} \leq |x| < 2^{-n}\})$ the intensity of Poisson process and by $F_n(\mathrm{d}x) = \frac{\nu(dx)}{\nu(\{x;2^{-(n+1)}\leq|x|<2^{-n}\})}$ the distribution of the jumps, with support on $\{x; 2^{-(n+1)} \leq |x| < 2^{-n}\}$. Then we can rewrite $\psi_3(k)$ as

$$\sum_{n\geq 0}\left[\lambda_n\int_{(2^{-(n+1)},2^{-n})}(e^{ikx}-1)F_n(\mathrm{d}x) - ik\lambda_n\Big(\int_{(2^{-(n+1)},2^{-n})}xF_n(\mathrm{d}x)\Big)\right]$$

The process $X^2(t)$ then corresponds to superposion of countable many compound Poisson processes. Sometimes is this part of the Lèvy process called *sum of compensated jumps*. Due to additional drift, the Lèvy measure on $(-1,1)$ is compensated. Without compesation, the limit in $X^2(t)$ may not converge as ε approaches 0.

4 Itô Formula

4.1 Quadratic Variation

Consider a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},P)$ and a semimartingale with càdlàg trajectories on it. The quadratic variation process is then defined as follows.

Definition 4.1 *Let $X = \{X(t), t\geq 0\}$ be a semimartingale. The* quadratic variation process $[X,X] = \{[X,X](t), t\geq 0\}$ *of X is defined by:*

$$[X,X](t) = X(t)^2 - 2\int_0^t X(s-)dX(s)$$

The quadratic variation process of X is càdlàg, increasing, adapted process. The starting value of the process is $[X,X](0) = X(0)^2$. The incremeants of the process are $\Delta[X,X](t) = (\Delta X(t))^2$.

More usefull from computational point of view is the following construction. Let $\pi(t) = \{0 = t_0 < t_1 < \ldots < t_k = t\}$ denotes the partition of the time interval $[0,t]$. Further consider the sequence of time partitions $\pi_n(t) = \{0 = t_0^n < t_1^n < \ldots t_{k_n}^n = t\}$ with supremum norm defined as $|\pi_n(t)| = \sup_{1\leq i\leq k_n}|t_i^n - t_{i-1}^n|$ and $\lim\limits_{n\to 0}|\pi_n(t)| = 0$. Then quadratic variation is the limit in probability of squared increments of the process X, uniformly continuous in time

$$\lim_{n\to\infty} P\Big(\omega\in\Omega : \sup_{0\leq s\leq t}\Big|X(0,\omega)^2 + \sum_{i=1}^{k_n}(X(t_i^n,\omega) - X(t_{i-1}^n,\omega))^2 - [X,X](t,\omega)\Big| > \varepsilon\Big) = 0$$

See Protter [4], Theorem 22, p.66.

The quadratic variation process can be decomposed into its continuous part and pure jump part. We denote the path-by-path continuous part of $[X,X]$ as $[X,X]^c$. We can then write the process in the following form

$$[X,X](t) = [X,X]^c(t) + X(0)^2 + \sum_{0<s\leq t}(\Delta X(s))^2 = [X,X]^c + \sum_{0\leq s\leq t}(\Delta X(s))^2$$

If the contiuous part is equal to zero, then

$$[X,X](t) = \sum_{0\leq s\leq t} (\Delta X(s))^2$$

4.1.1 Quadratic Variation for Stable Processes

Let us formulate the quadratic variation process for a Lèvy process. Consider Lèvy process with characteristic triplet (σ,ν,γ) Then the quadratic variation is given by

$$[X,X](t) = \sigma^2 t + \sum_{0<s\leq t} (\Delta X(s))^2 = \sigma^2 t + \int_0^t \int_{\mathbb{R}\setminus 0} x^2 N(dt,\cdot,dx)$$

Consider $\alpha-$stable Lèvy motion $L_{\alpha,\beta} = \{L_{\alpha,\beta}(t), t\geq 0\}$. Then for $\alpha = 2$ the quadratic variation process is a continuous increasing process, whereas for $0<\alpha<2$ the quadratic variation is the sum of squared jumps.

$$[X,X](t) = \begin{cases} \sum\limits_{0\leq s\leq t} (\Delta X(s))^2 = \int_0^t \int_{\mathbb{R}\setminus 0} x^2 N(dt,\cdot,dx) & \text{for } 0<\alpha<2 \\ \sigma^2 t & \text{for } \alpha = 2 \end{cases}$$

4.2 Change of Variable Formula

Let us start with the general version of change of variable formula for semimartingales, i.e. situation when one needs to map the semimartingale via the twice differentiable continuous function. Consider filtered probability space $(\Omega,\mathcal{F},\mathbb{F},P)$ and let $X = \{X(t), t\geq 0\}$ denote a semimartingale with càdlàg trajectories. The Itô's Formula has to capture also discontinuity term and is of following form:

Theorem 4.1 *Let $X = \{X(t), t\geq 0\}$ be a semimartingale and let $f \in \mathcal{C}^2(\mathbb{R})$. Then $f(X)$ is again a semimartingale, and the following formula holds:*

$$\begin{aligned} f(X(t)) - f(X(0)) = \int_{0+}^t f'(X(s-))dX(s) + \frac{1}{2}\int_{0+}^t f''(X(s-))d[X,X]^c(s) + \\ + \sum_{0<s\leq t} \Big(f(X(s)) - f(X(s-)) - f'(X(s-))\Delta X(s) \Big) \end{aligned} \tag{11}$$

Remark 4.1 *The second term on the right side is given by*

$$f''(X(t-))d[X,X](t) = f''(X(t-))[X,X]^c(t) + f''(X(t-))(\Delta X(t))^2$$

and so the relation in (11) can be equivalently written as

$$\begin{aligned} f(X(t)) - f(X(0)) = \int_{0+}^t f'(X(s-))dX(s) + \frac{1}{2}\int_{0+}^t f''(X(s-))d[X,X]^c(s) + \sum_{0<s\leq t} f''(X(s-))(\Delta X(s))^2 + \\ + \sum_{0<s\leq t} \Big(f(X(s)) - f(X(s-)) - f'(X(s-))\Delta X(s) - f''(X(s-))(\Delta X(s))^2 \Big) \end{aligned}$$

Consider now a Lèvy process $X = \{X(t), t \geq 0\}$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Lèvy process has generating triplet (σ, ν, γ). Further consider $f \in \mathcal{C}^2(\mathbb{R})$ and process $Y = \{Y(t) := f(X(t)), t \geq 0\}$. Applying Theorem 4.1 and using notation introduced in section 3.1 we derive

$$Y(t) = f(X(0)) + \int_0^t f'(X(s-))dX(s) + \frac{1}{2}\int_0^t \sigma^2 f''(X(s-))\,\mathrm{d}s + \\ + \int_0^t \Big(f(X(s-)+z) - f(X(s-)) - zf'(X(s-))\Big)N(\mathrm{d}s,\cdot,\mathrm{d}z)$$

Recall now Lèvy-Itô decomposition and rewrite $dX(s)$ as $dX^1(s) + dX^2(s) + dX^3(s)$.

$$Y(t) = f(X(0)) + \int_0^t \gamma f'(X(s))\,\mathrm{d}s + \int_0^t \sigma^2 f'(X(s))dW(s) + \\ + \int_0^t \int_{\{0<|z|<1\}} zf'(X(s-))\tilde{N}(\mathrm{d}s,\cdot,\mathrm{d}z) + \int_0^t \int_{\{1\leq|z|<\infty\}} zf'(X(s-))N(\mathrm{d}s,\cdot,\mathrm{d}z) + \\ + \frac{1}{2}\int_{0+}^t f''(X(s-))\sigma^2\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}\setminus 0} \Big(f(X(s-)+z) - f(X(s-)) - zf'(X(s-))\Big)N(\mathrm{d}s,\cdot,\mathrm{d}z)$$

Recall that $\tilde{N}(\mathrm{d}t,\cdot,\mathrm{d}x) = N(\mathrm{d}t,\cdot,\mathrm{d}x) + \nu(\mathrm{d}x)\,\mathrm{d}t$ and reorder the equation.

$$Y(t) = f(X(0)) + \int_0^t \sigma^2 f'(X(s))dW(s) + \int_0^t \int_{\mathbb{R}\setminus 0} zf'(X(s-))\tilde{N}(\mathrm{d}s,\cdot,\mathrm{d}z) + \\ + \int_0^t \Big[\gamma f'(X(s)) + \frac{1}{2}f''(X(s-))\sigma^2 + \int_{\mathbb{R}\setminus 0}\Big(f(X(s-)+z) - f(X(s-)) - zf'(X(s-))1_{\{0<|x|<1\}}\Big)\nu(\mathrm{d}z)\Big]\,\mathrm{d}s$$

Theorem 3.2 states that we can decompose Lèvy process into martingale with bounded jumps and process with paths of finite variation. For the considered process Y we see that process with paths of finite variation is

$$V(t) = \int_0^t \Big[\gamma f'(X(s)) + \frac{1}{2}f''(X(s-))\sigma^2 + \int_{\mathbb{R}\setminus 0}\Big(f(X(s-)+z) - f(X(s-)) - zf'(X(s-))1_{\{0<|x|<1\}}\Big)\nu(\mathrm{d}z)\Big]\,\mathrm{d}s$$

and the martingale part

$$M(t) = f(X(0)) + \int_0^t \sigma^2 f'(X(s))dW(s) + \int_0^t \int_{\mathbb{R}\setminus 0} zf'(X(s-))\tilde{N}(\mathrm{d}s,\cdot,\mathrm{d}z)$$

Using the previous results let us introduce the *Itô-Lèvy process* and fomulate the change of variable formula for it in differential notation. Consider again a Lèvy process $X = \{X(t), t \geq 0\}$ and predictable processes $u = \{u(t), t \geq 0\}$ $v = \{v(t), t \geq 0\}$ and $w = \{w(t), t \geq 0\}$. From the Lèvy-Itô decomposition we call X the *Itô-Lèvy process* if it is given as follows

$$X(t) = x_0 + \int_0^t u(s)ds + \int_0^t v(s)dW(s) + \int_0^t \int_{\mathbb{R}\setminus 0} w(s,x)\tilde{N}(ds,\cdot,dx),$$

where for all $t > 0, x \in \mathbb{R}\setminus 0$

$$\int_0^t \Big(|u(s)| + v^2(s) + \int_{\mathbb{R}\setminus 0} w^2(s,x)\nu(dx)\Big)ds < \infty, \qquad P-a.s.$$

This condition implies that the stochastic integrals are well-defined and local martingales.
One uses the short-hand differential notation:

$$dX(t) = u(t)dt + v(t)dW(t) + \int_{\mathbb{R}\setminus 0} w(t,x)\tilde{N}(dt,dx);\ X(0) = x_0$$

Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a function in $\mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and define process:

$$Y(t) := f(t, X(t)), \quad t \geq 0.$$

Then the process $Y = \{Y(t), t \geq 0\}$ is also an Itô-Lèvy process and its differential form is given by

$$\begin{aligned}
dY(t) &= \frac{\partial f}{\partial t}(t,X(t))dt + \frac{\partial f}{\partial x}(t,X(t))u(t)dt + \\
&+ \frac{\partial f}{\partial x}(t,X(t))v(t)dW(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X(t))v^2(t)dt + \\
&+ \int_{\mathbb{R}\setminus 0}[f(t,X(t)+w(t,z)) - f(t,X(t)) - \frac{\partial f}{\partial x}(t,X(t))w(t,z)]\nu(dz)dt + \\
&+ \int_{\mathbb{R}\setminus 0}[f(t,X(t-)+w(t,z)) - f(t,X(t-))]\tilde{N}(dt,dz)
\end{aligned}$$

4.3 Stochastic Exponential Driven by Lèvy process

In this part we investigate the simple example of the stochastic differential equation which solution is called *Doleéns-Dade exponential*.

Theorem 4.2 *Let $X = \{X(t), t \geq 0\}$ be a Lèvy process with characteristic triplet (σ, ν, γ). Then there exists a càdlàg process $Z = \{Z(t), t \geq 0\}$ that is the unique solution to the equation*

$$Z(t) = 1 + \int_0^t Z(s-)dX(s)$$

Z is called stochastic exponential *of X, denoted by $Z = \mathcal{E}(X)$ and is expressed by*

$$Z(t) = \exp\left\{X(t) - \frac{\sigma^2 t}{2}\right\} \prod_{0<s\leq t} (1 + \Delta X(s)) \exp\left\{-\Delta X(s) + \frac{1}{2}(\Delta X(s))^2\right\}$$

where the infinite product converges.

The above stated result holds for semimartingales in general, see Protter [4], II.8,p.84.

If X is $\alpha-$stable Lèvy motion $L_{\alpha,\beta} = \{L_{\alpha,\beta}(t), t \geq 0\}$, the stochastic exponential Z is

$$Z(t) = \begin{cases} \prod_{0<s\leq t}(1 + \Delta L_{\alpha,\beta}(s)) \exp\left\{-\Delta L_{\alpha,\beta}(s) + \frac{1}{2}(\Delta L_{\alpha,\beta}(s))^2\right\} & \text{for } 0 < \alpha < 2 \\ \exp\left\{W(t) - \frac{\sigma^2 t}{2}\right\} & \text{for } \alpha = 2 \end{cases}$$

where $W(t)$ denotes Brownian motion.

The stochastic exponential driven by $\alpha-$stable Lèvy motion is always positive for $\alpha = 2$ and corresponds to geometric Brownian motion. On the other hand if stability parameter α has value from $(0, 2)$, the stochastic exponential can take even negative values. Important role plays obviously initial

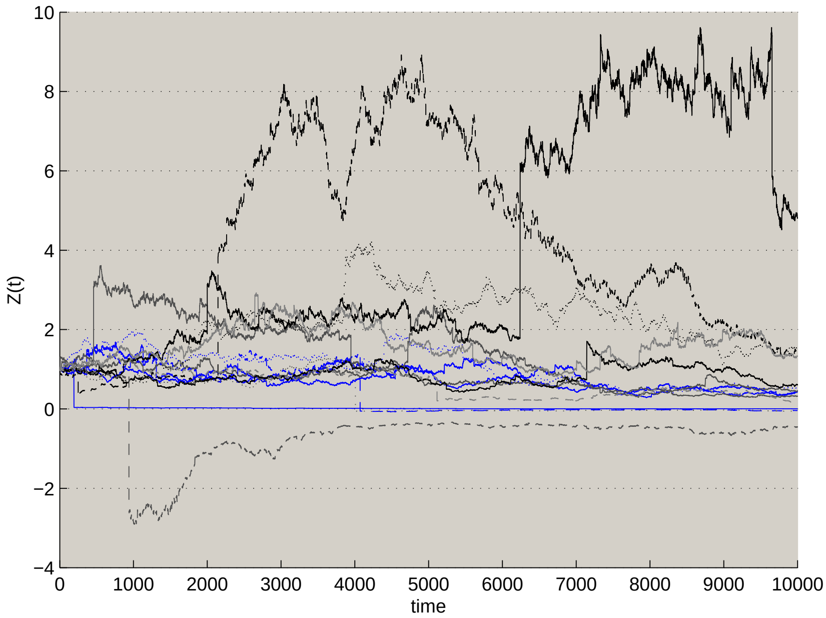

Figure 1: 15 trajectories of stochastic exponential driven by $\alpha-$stable Lèvy motion with parameters $\alpha = 1.7$, and $\beta = 0.3$

value of stochastic exponential Z and size of jumps of the process. In our case $Z(0) = 1$. If the sizes of the jumps of $L_{\alpha,\beta}$ are biger then the initial value 1, stochastic exponential can take negative values.

It is possible to simulate trajectory of the stochastic exponential driven by $\alpha-$stable Lèvy motion according the algorithm outlined in Janicki et al. [3]. The main idea is that the solution of the equation:

$$Z(t) = Z_0 + \int_0^t Z(s-)dL_{\alpha,\beta}$$

can be aproximated by

$$Z_n(t) = Z_{n,0} \prod_{k=1}^{[nt]} \left(1 + \frac{Y_k}{\phi(n)}\right)$$

where $Y_k, k = 1, 2, \ldots n$ are i.i.d sequence of $Y_k \sim S_\alpha(1, \beta, 0)$ and $\phi(n)$ is slowly varying function which is choosen as $\phi(n) = n^{1/\alpha}$.

In the following figures we depicted trajectories of stochastic exponential driven by $\alpha-$stable Lèvy motion for different stability parameters. The observation which one can make is that with stability parameter closer to 2, the process has less big jumps. Also important observation is that if the increment of the process is very close to or exactly the initial value, stochastic exponential can then remain very close to 0.

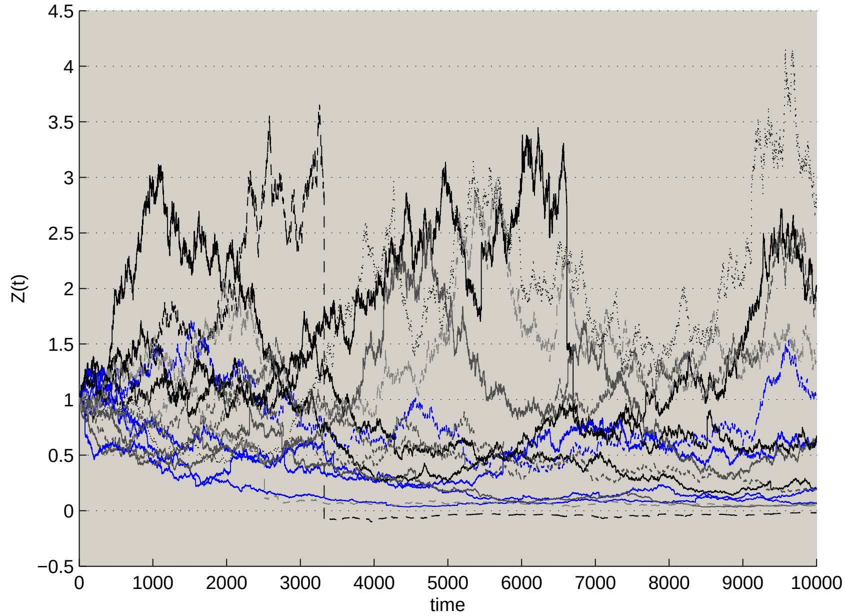

Figure 2: 15 trajectories of stochastic exponential driven by $\alpha-$stable Lèvy motion with parameters $\alpha = 1.9$, and $\beta = 0.3$

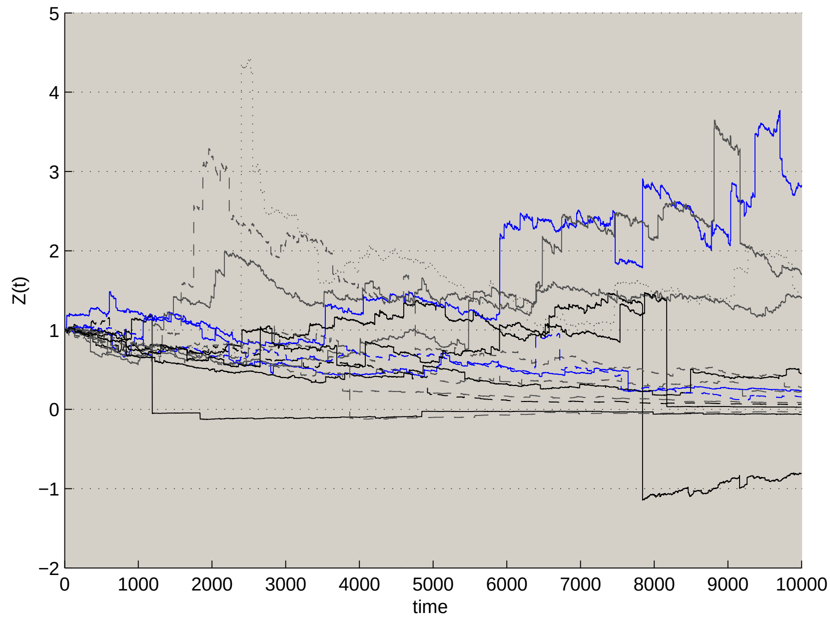

Figure 3: 15 trajectories of stochastic exponential driven by $\alpha-$stable Lèvy motion with parameters $\alpha = 1.3$, and $\beta = 0.3$

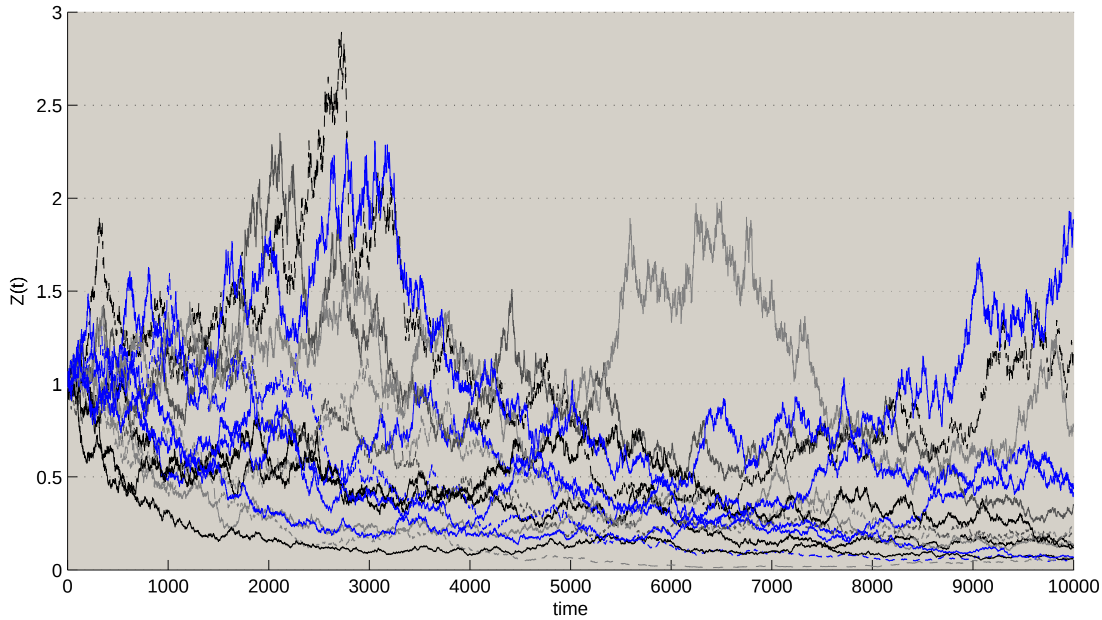

Figure 4: 15 trajectories of stochastic exponential driven by $\alpha-$stable Lèvy motion with parameters $\alpha = 2$, and $\beta = 0$, i.e. geometric Brownian motion

5 Change of Measures

Consider measurable space $(\Omega, \mathcal{F})$ with two probability laws P and Q. Further assume that P and Q are *equivalent* probability measures, i.e. $P \ll Q$ and $Q \ll P$, where $P \ll Q$ means that measure P is absolutly continuous with respect to measure Q:

$$\forall F \in \mathcal{F} : Q(F) = 0 \implies P(F) = 0$$

The equivalence of the measures is also sometimes called *mutual absolut continuousness* of measures P and Q. We write $P \sim Q$ to denote equivalence.

Let us endow the measurable space $(\Omega, \mathcal{F})$ with two equivalent probability measures P, Q and consider two probability spaces $(\Omega, \mathcal{F}, P)$ and $(\Omega, \mathcal{F}, Q)$. We know that possible events can occur on these spaces with same or different non-zero probabilities. By assuming the equivalence of the measures, we are, however, ensured that the events which occur with probability zero in one space will remain impossible also in the other probability space. In other words measures P and Q have the same support or equivalently same null sets.

Consider restriction of the probability measure P to $\mathcal{F}_t$ and denote it as $P_t = P|\mathcal{F}_t$. Analogically denote $Q_t = Q|\mathcal{F}_t$. As the measures P, Q are considered on the same stochastic bases, i.e. measureable space with the filtration, then also the restrictions P_t, Q_t of the equivalent measures P, Q at time t remains equivalent for all $t \geq 0$. The Radon-Nikodym theorem ensures us that at every time moment t there exists a measurable mapping $D(t)$, s.t. $Q_t = \int_A D(t)\, \mathrm{d}P_t$ for all $A \in \mathcal{F}_t$. The stochastic process $D = \{D(t) = \frac{\mathrm{d}Q_t}{\mathrm{d}P_t}, t \geq 0\}$ is called *derivative process* of Q with respect to P. Derivative process describes the time evolution of the density of measure Q with respect to measure P and filtration.

Obviously, the derivative process depends on the choice of the filtration. Different filtrations induce different derivative processes.

Let $X = \{X(t), t \geq 0\}$ be a Lèvy process on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. We would like to create new Lèvy process on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$ from the old Lèvy process X.

- Under which conditions the new stochastic process on $(\Omega, \mathcal{F}, \mathbb{F}, Q)$ remains also Lèvy process ?
- Can we somehow remove the drift part of the Lèvy process and create a martingale?
- What does happen to jump structure of the Lèvy process when we pass from one probability space to the other?
- How can one describe the derivative process and is the derivative process also Lèvy process ?

Let us denote by (X, P) the Lèvy process with generating triplet $(\sigma_P, \nu_P, \gamma_P)$ on filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Analogically, we denote (X, Q) as the Lèvy process with generating triplet $(\sigma_Q, \nu_Q, \gamma_Q)$ on the second considered space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$.

The following theorem from Sato [5],33.1,p.218 gives a necessary and sufficient condition for equivalence of the measures in term of generating triplets of the Lèvy processes.

Theorem 5.1 *Let (X, P) and (X, Q) be Lèvy processes on $\mathbb{R}$ with generating triplets $(\sigma_P, \nu_P, \gamma_P)$ and $(\sigma_Q, \nu_Q, \gamma_Q)$, respectively. Then the following statements are equivalent:*

1. *$P_t \sim Q_t$ for every $t > 0$*
2. *the generating triplets satisfy*

$$\sigma_P = \sigma_Q, \tag{12}$$

$$\nu_P \sim \nu_Q \tag{13}$$

with the function $\phi(x)$ defined by $\frac{d\nu_Q}{d\nu_P} = e^{\phi(x)}$ satisfying

$$\int_{\mathbb{R}} (e^{\phi(x)/2} - 1)^2 \nu_P(dx) < \infty \tag{14}$$

and

$$\gamma_Q - \gamma_P - \int_{|x| \leq 1} x(\nu_Q - \nu_P)(dx) \in \{\sigma_P^2 x : x \in \mathbb{R}\} \tag{15}$$

If we work only with drifted diffusion, we see that the only limiting condition for us is to have same diffusive coefficient. The assumption on the finitiness of the difference of the drift parameter is rather natural. In case of diffusion we can freely change the drift. Now consider only pure jump process. There we are limited on the behaving of the small jumps in terms of drift. The expected value of the small jumps (on the unit ball) measured by the difference of the Lèvy measures has to be equal to difference of the drifts

The following theorem from Sato [5],33.2 p.219 gives exact represantation for the derivative process:

Theorem 5.2 *Let (X,P) and (X,Q) be Lèvy processes on $\mathbb{R}$ with generating triples $(\sigma_P,\nu_P,\gamma_P)$ and $(\sigma_Q,\nu_Q,\gamma_Q)$, respectively. Suppose that equivalent conditions 1. and 2. from previous theorem are satisfied. Choose $\eta\in\mathbb{R}$ such that*

$$\gamma_Q-\gamma_P-\int_{|x|\leq 1} x(\nu_Q-\nu_P)(dx)=\sigma_P^2\eta \tag{16}$$

Then we can define, $P-a.s$,

$$U(t)=\eta X^c(t)-\frac{\eta^2\sigma_P^2 t}{2}-\eta\gamma_P t+ \tag{17}$$

$$+\lim_{\varepsilon\to 0^+}\left(\sum_{\substack{s\leq t,\\ |\Delta X(s)|>\varepsilon}}\phi(\Delta X(s))-t\int_{|x|>\varepsilon}(e^{\phi(x)}-1)\nu_P(dx)\right) \tag{18}$$

The convergence in the right-hand side of (6) is uniform in t on any bounded interval, $P-a.s$. We have, for every $t\geq 0$

$$\mathbb{E}^P e^{U(t)}=\mathbb{E}^Q e^{-U(t)}=1 \tag{19}$$

and

$$\frac{\mathrm{d}Q_t}{\mathrm{d}P_t}=e^{U(t)},\ P-a.s. \tag{20}$$

The process $(\{U(t),t\geq 0\},P)$ is a Lèvy process on $\mathbb{R}$ with generating triplet $(\sigma_U,\nu_U,\gamma_U)$ expressed by

$$\sigma_U^2=\eta^2\sigma_P^2 \tag{21}$$

$$\nu_U=[\nu_P\phi^{-1}]_{\mathbb{R}-\{0\}} \tag{22}$$

$$\gamma_U=-\frac{\eta^2\sigma_P^2}{2}-\int_{\mathbb{R}}(e^y-1-y1_{\{0<|y|\leq 1\}}(y))(\nu_P\phi^{-1})(dy) \tag{23}$$

5.1 Equivalence of Measures for Stable Processes

Let us investigate if it is possible construct equivalent measure for $\alpha-$stable Lèvy motion .

Consider filtered probability spaces $(\Omega,\mathcal{F},\mathbb{F},P)$ and $(\Omega,\mathcal{F},\mathbb{F},Q)$. Denote $\alpha-$stable Lèvy motion on $(\Omega,\mathcal{F},\mathbb{F},P)$ as $(L_{\alpha,\beta},P)$ with generating triplet $(0,\nu_P,\gamma_P)$ for $\alpha\in(0,2)$ and $(\sigma_P,0,\gamma_P)$ for $\alpha=2$. Analogically, we denote $\alpha-$stable Lèvy motion on $(\Omega,\mathcal{F},\mathbb{F},Q)$ as $(L_{\alpha,\beta},Q)$ with generating triplet $(0,\nu_Q,\gamma_Q)$ for $\alpha\in(0,2)$ and $(\sigma_Q,0,\gamma_Q)$ for $\alpha=2$.

The first quick observation which one can make for $\alpha-$stable Lèvy motion is that if we have diffusive process under measure P, the process under measure Q has to be also diffusive. If we have pure jump process under measure P then also new process will be pure jump under measure Q. It means that we definitly cannot create pure jump process from diffusive process and vice verse.

Let us first investigate simplier case for $\alpha=2$. Using Theorem 5.1 we state under which conditions are measures P and Q equivalent. One neccesary assumption is on diffusion terms as these have to be equal, i.e. $\sigma_P=\sigma_Q$. The second assumption gives us that the difference of drift terms $\gamma_Q-\gamma_P$ has to be finite number. If these conditions holds, then P_t and Q_t will be equivalent measures for all $t\geq 0$.

We can now use Theorem 5.2 to exactly describe the derivative process. We choose $\eta \in \mathbb{R}$ s.t. $\gamma_Q - \gamma_P = \sigma_P^2 \eta$ and from there we can write

$$\eta = \frac{\gamma_Q - \gamma_P}{\sigma_P^2} \tag{24}$$

To simplify the notation we denote $(L_{2,0}, P)$ 2-stable symmetric Lèvy motion as (W, P), i.e. drifted Brownian motion with drift parameter γ_P. Analogically, we redenote $(L_{2,0}, Q)$ as (W, Q). Using the result from Theorem 5.2, expression (17) for η and continuousness of trajectories of Brownian motion, we easily obtain stochastic process $U(t)$, the logarithm of derivative process $\frac{dP}{dQ}$, $P - a.s.$, in the following form:

$$U(t) = -\frac{(\gamma_Q - \gamma_P)^2}{2\sigma_P^2} t + \frac{\gamma_Q - \gamma_P}{\sigma_P^2} W(t) \tag{25}$$

$U = \{U(t), t \geq 0\}$ is a Lèvy process on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with generating triplet $\left(\frac{\gamma_Q - \gamma_P}{\sigma_P^2}, 0, -\frac{(\gamma_Q - \gamma_P)^2}{2\sigma_P^2}\right)$ which corresponds to results of Theorem 5.2. The derivative process $D = \{D(t), t \geq 0\}$ is then the exponential of U and also Dolean-Dade exponential,

$$D(t) = \exp\left\{ -\frac{(\gamma_Q - \gamma_P)^2}{2\sigma_P^2} t + \frac{\gamma_Q - \gamma_P}{\sigma_P^2} W(t) \right\} \tag{26}$$

These results for 2-stable Lèvy motion are not very surprising as we could derive the same very nicely and moreove claim even more also about the martingale property by using the famous Girsanov theorem.

The more interesting for us is then the case for $\alpha \in (0,2)$. We obtain rather surprising result which we formulate into following corollary. Consider two filtered probability spaces $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and $(\Omega, \mathcal{F}, \mathbb{F}, Q)$.

Theorem 5.3 *Let $(L_{\alpha_P, \beta_P}, P)$ and $(L_{\alpha_Q, \beta_Q}, Q)$ be $\alpha-$stable Lèvy motions on $\mathbb{R}$ with parameters $\alpha_P, \alpha_Q \in (0,2)$, $\beta_P, \beta_Q \in [-1,1]$ and drift terms $\gamma_P, \gamma_Q \in \mathbb{R}$. Then the probability measures P_t and Q_t are equivalent, for all $t \geq 0$, if and only if at once $\alpha_P = \alpha_Q$ and $\beta_P = \beta_Q$ and $\gamma_P = \gamma_Q$.*

Contrary to case when $\alpha = 2$ where we have possibility to change and if necessary completly remove the drift term of $\alpha-$stable Lèvy motion , in pure jump case we can neither change the stability parameter, nor the skewness of the distribution of the increments of the process. This implies that we can neither change the behaving of small jumps nor big jumps of the process and further we even cannot alter the drift part of the process.

Proof 1 *Consider $\alpha-$stable Lèvy motion $(L_{\alpha_P, \beta_P}, P)$, $(L_{\alpha_Q, \beta_Q}, Q)$ with characteristic triplets $(0, \nu_P, \gamma_P)$, $(0, \nu_P, \gamma_P)$ respectively.*
γ_P and γ_Q are real numbers and Lèvy measures are of the form

$$\nu_P(\mathrm{d}x) = \frac{c_P^+}{x^{\alpha_P+1}} 1_{\{x>0\}} + \frac{c_P^-}{|x|^{\alpha_P+1}} 1_{\{x<0\}} \qquad \textit{for } 0 < \alpha_P < 2 \ \textit{ and } c_P^+, c_P^- \geq 0, c_P^+ + c_P^- > 0$$
$$\nu_Q(\mathrm{d}x) = \frac{c_Q^+}{x^{\alpha_Q+1}} 1_{\{x>0\}} + \frac{c_Q^-}{|x|^{\alpha_Q+1}} 1_{\{x<0\}} \qquad \textit{for } 0 < \alpha_Q < 2 \ \textit{ and } c_Q^+, c_Q^- \geq 0, c_Q^+ + c_Q^- > 0$$

We now use Theorem 5.1 to state conditions on stability and skewness parameters. Without lost of generality we can work only on the positive half-axis of the real line. For this reason we simplify the notation for Lèvy measures

$$\nu_P(\mathrm{d}x) = \frac{c_P}{x^{\alpha_P+1}} \qquad \text{for } 0 < \alpha_P < 2 \text{ and } c_P \geq 0, x > 0$$

$$\nu_Q(\mathrm{d}x) = \frac{c_Q}{x^{\alpha_Q+1}} \qquad \text{for } 0 < \alpha_Q < 2 \text{ and } c_Q \geq 0, x > 0$$

1. *Measures ν_P and ν_Q have to have same support. This holds if at once $c_P > 0$ and $c_Q > 0$ or $c_P = 0$ at the same time $c_Q = 0$.*
2. *Further, for Lèvy measures must hold the condition on the finiteness of the Hellinger distance between these measures, stated as:*
   consider function $\phi(x)$ defined as the logarithm of density of Lèvy measure ν_Q with respect to Lèvy measure ν_P, i.e. $\frac{\mathrm{d}\nu_Q}{\mathrm{d}\nu_P} = e^{\phi(x)}$, such that
   $$\int_0^\infty (e^{\phi(x)} - 1)^2 \nu_P(\mathrm{d}x) < \infty$$
   The function $\phi(x)$ is then
   $$\phi(x) = \log\left(\frac{c_P}{c_Q} x^{(\alpha_P - \alpha_Q)}\right)$$
   which states further condition on $c_P > 0$ and from previous even $c_Q > 0$.
   We need to investigate finiteness of the integral
   $$c_P \int_0^\infty \left(\sqrt{\frac{c_Q}{c_P}} x^{(\alpha_P - \alpha_Q)/2} - 1\right)^2 \frac{\mathrm{d}x}{x^{1+\alpha_P}}$$
   The latter can be rewriten as
   $$\int_0^\infty \left(\frac{\sqrt{c_Q}}{x^{(\alpha_Q+1)/2}} - \frac{\sqrt{c_P}}{x^{(\alpha_P+1)/2}}\right)^2 \mathrm{d}x = \int_0^\infty \left(\frac{c_Q}{x^{(\alpha_Q+1)}} + \frac{c_P}{x^{(\alpha_P+1)}} - \frac{2\sqrt{c_P c_Q}}{x^{(\alpha_P+\alpha_Q)/2}}\right) \mathrm{d}x$$
   Consider first that $\alpha_P \neq \alpha_Q$. The integral diverges and the Hellinger distance of these measure is thus not finite. The choice c_P and c_Q does not play a role in this situation.
   Condsider then that $\alpha_P = \alpha_Q$. The integral can be rewritten into simplier form
   $$\int_0^\infty \frac{(\sqrt{c_P} - \sqrt{c_Q})^2}{x^{\alpha+1}} \mathrm{d}x$$
   From there is obvious that the intergal will be finite only when $c_P = c_Q$.
   The conclusion is that measures ν_P and ν_Q has to be identical.
3. *The last condition in Theorem 5.1 is the restriction on the drift term. From absence of Gaussian term and equality of the Lèvy measures follows that $\gamma_Q = \gamma_P$.*

The last which needs to be checked is the equality of skewness parameters $\beta_P = \beta_Q$. It follows easily from the relationship between skewness parameter β_P and parameters of Lèvy measure c_P^+, c_P^-. Recall that $\beta_P = \frac{c_P^+ - c_P^-}{c_P^+ + c_P^-}$ and $c_P^+ = c_Q^+, c_P^- = c_Q^-$.

Q.E.D

We investigated the situation only for $\alpha-$stable Lèvy motion on $\mathbb{R}$. It would need to be checked and proved if the same situation holds also for $\mathbb{R}^d$ with $d \geq 2$.

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